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Certain inequalities for (α,β) - normal transaloid operators on Hilbert spaces

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ABSTRACT

Let H be an infinite dimensional complex Hilbert space and $N_T(H)$ denote the class of (α,β)-normal transaloid operators. In this paper, we characterize (α,β)- normal transaloid operators and later determine norm and numerical radii inequalities for (α,β)-normal transaloid operators.

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1. Introduction

A lot of researchers and mathematicians have done work extensively on various inequalities for normal operators in Hilbert space, for instance on numerical ranges, numerical radius, convexoid, normaloid, spectraloid, self adjoint and other classes of non-normal operators in operator theory. However, Dragomir and Moslehian (Dragomir & Moslehian, 2008) recently worked on various inequalities between norm and numerical radius of (α,β) - normal operators. Senthilkumar (Senthilkumar D., 2014) studied generalization of p-(α,β) - normal operators but not for normal transaloid operators. In this paper we characterize (α,β)-normal transaloid operators. We consider the properties of this class of operators and give results on their characteristics. The properties which have been considered include: linearity, continuity, positivity, boundedness, reflexivity and self-adjointedness. Subsequently, we reviewed some basic concepts and definitions which are useful to our work as outlined in the section below.

2. Preliminaries

Definition 2.1 Let X be a linear space. A non-negative real valued function $\|.\|: X \to \mathbf{R}$ is called a norm on X if it satisfies the following properties for all $x, y \in X$:

i. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ (positivity).
ii. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in C$ (Homogeneity)
iii. $\|x + y\| = \|x\| + \|y\|$ (Triangle inequality)

Definition 2.2 An operator is a structure preserving maps. For example; let V be a vector space, a non-negative real valued function $T: V \to V$ then T is considered to be an operator.

Definition 2.3 Let X be a vector space over **K.** A non-negative real valued function $\langle .,. \rangle : X \times X \to \mathbf{K}$ is called an inner product space if for $x, y \in X$ and $\alpha \in \mathbf{K}$ satisfies the following properties;

i. $\langle x, x \rangle \geq 0$, for all $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
ii. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.
iii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

The pair $(X, \langle .,. \rangle)$ is called an inner product space.

Definition 2.4 Let $T: H \to H$ be a linear operator. Then T is said to be normal if $TT^* = T^*T$

Definition 2.5 Let $T \in B(H, S)$ where H and S are Hilbert spaces, then the linear operator $T^* \in (S, H)$ satisfies $\langle Tx, y \rangle = \langle x, T^* y \rangle$, for all $x \in H$ and $y \in S$ is called an adjoint operator of T.

Definition 2.6 Let $T: H \to H$ be a linear operator, then an operator T bounded if there exist a constant $M \geq 0$ such that: $\|Tx\| \leq M\|x\|$, for all $x \in H$.

Definition 2.7 An operator T acting on a Hilbert space is called (α,β)-normal $(0\leq\alpha\leq 1\leq\beta)$ if $\alpha^2T^*T\leq TT^*\leq\beta^2T^*T$

Definition 2.8 Let $T:H\rightarrow H$ be a linear operator, then T is normaloid if $\|T\|=w(T)$ and transaloid if $T=T-\lambda I$ for all $\lambda\in\mathbf{C}$.

3. Main Results

Lemma 3.1. Let A be in $N_T(H)$ then A is self adjoint.

Proof: Let H be a complex Hilbert space, and let $A\in N_T(H)$. We need to show that $A=A^*$. Indeed, for any $z\in H$. We can have $\overline{\langle Az,z\rangle}=\langle z,Az\rangle=\langle A^*z,z\rangle=\langle Az,z\rangle$. Therefore, $\langle Az,z\rangle$ is real. Also assume that $\langle Az,z\rangle$ is real for all z, then choose any $z,g\in H$. We can have $\langle A(z+g),z+g\rangle=\langle Az,z\rangle+\langle Az,g\rangle+\langle Ag,z\rangle+\langle Ag,g\rangle$ this implies that $\langle A(z+g),z+g\rangle,\langle Az,z\rangle,\langle Ag,g\rangle$ are real, we can conclude that $\langle Az,g\rangle+\langle Ag,z\rangle$ is real and it equals its own complex conjugate, for instance

$$\langle Az,g\rangle+\langle Ag,z\rangle=\overline{\langle Az,g\rangle+\langle Ag,z\rangle}=\langle g,Az\rangle+\langle z,Ag\rangle \quad \text{...(1)}$$

We can see that

$i\langle Az,g\rangle+i\langle Ag,z\rangle=\overline{i\langle Az,g\rangle+i\langle Ag,z\rangle}=i\langle g,Az\rangle-i\langle z,Ag\rangle$ by multiplying i throughout, we can have

$$\langle Az,g\rangle-\langle Ag,z\rangle=-\langle g,Az\rangle+\langle z,Ag\rangle \quad \text{...(2)}$$

By combining (1) and (2) we can obtain $2\langle Az,g\rangle=2\langle z,Ag\rangle=2\langle A^*z,g\rangle$. Therefore, this is true for z and g, we can conclude that $A=A^*$. Hence A is a self adjoint.

Theorem 3.2 Let A be in $N_T(H)$ then A is reflexive, bounded, linear and continuous.

Proof: Linearity; Let $A\in N_T(H)$. Let $x_1,x_2\in DomA$ and $\alpha,\beta\in K$ then there exist V of $DomA$ such that $x_1,x_2\in V$ then $(\alpha_1x_1+\beta_2x_2,\alpha_1Af_1+\beta_2Af_2)\in V$. But by definition of A it is clear that $A(\alpha_1x_1+\beta_2x_2)=\alpha_1Ax_1+\beta_2Ax_2$. Hence A is linear.

Boundedness; Suppose that H_1 and H_2 are Hilbert spaces over $\mathbf{K}$. Then a linear operator $A:H_1\rightarrow H_2$ is bounded if there exist a real number $M\geq 0$ such that $\|Ax\|\leq M\|x\|$ $\forall x\in H_1$.

Consider now a linear operator $A: H_1 \to H_2$, where H_2 is a Hilbert space over **K** such that we can have

$\|A(x)\| = \|A(c_1 v_1 + \ldots + c_n v_n)\| \leq \sum_{i=1}^{n} \|c_i A(v_i)\| = \sum_{i=1}^{n} |c_i| \|A(v_i)\|$. Using the Cauchy Schwarz inequality, we have $\sum_{i=1}^{n} |c_i| \|A(v_i)\| = (\sum_{i=1}^{n} |c_i|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} \|A(v_i)\|^2)^{\frac{1}{2}}$. It follows that $\|A(x)\| \leq (\sum_{i=1}^{n} \|A(x)\|^2)^{\frac{1}{2}} \|x\|$ and so the linear operator $A: H_1 \to H_2$ is bounded with $\|Ax\| \leq M\|x\| \quad \forall x \in H_1$. Similarly, a linear operator $A: H_1 \to H_2$ is continuous if and only if it is bounded.

Continuity: Suppose A is bounded. Then there exist $M \geq 0$ such that $\|A(x) - A(y)\| = \|A(x-y)\| \leq M\|x-y\| \quad \forall x, y \in H_1$ and consequently A is continuous. Now suppose A is continuous. Obviously $A(0) = 0$. Then for $\varepsilon = 1$ there exist $\delta > 0$ such that $\|Ax\| < \varepsilon = 1 \quad \forall \|x\| < \delta$. For $v \in H_1, v \neq 0$ then $A(v) = \frac{2\|v\|}{\delta} A(\frac{\delta}{2\|v\|} v)$. Since $\left\| \frac{\delta}{2\|v\|} v \right\| = \frac{\delta}{2} < \delta$ we get $\|A(v)\| \leq \frac{2\|v\|}{\delta}$ and consequently A is bounded.

Reflexivity; We need to consider V and $V^{\sim}$ to be finite dimensional subspaces of $N_T(H)$, then V is algebraically reflexive if the $\dim(V^{\sim}|_R)$ is finite. By (Dragomir & Moslehian., 2008, Theorem 4.7) it is sufficient enough to show that $\dim(V^{\sim}|_R)$ is algebraically bounded reflexive since bounded reflexive implies algebraic bounded reflexivity, hence $\dim(V^{\sim}|_R)$ is bounded reflexive. Therefore, let $W \in ref_b(V^{\sim}|_R)$ and Let $W^{\sim}$ be the extension of W to $[R]$, where $[R]$ is the norm closure of R. By $W \in ref_b(V^{\sim}|_R)$, there exists R_W such that for any $a \in R$, $W_a \in [V^{\sim}|_{RW}\, a]$. Then $\dim(V^{\sim}|_R)$ is finite. Since R contains separating vector of $V^{\sim}$, if V is a finite dimensional subspace of $N_T(H)$ with a separating vector then V is boundedly reflexive. Moreover, we can have $W \in V^{\sim}|_{[R]}$. Hence $W \in V^{\sim}|_R$ and $V^{\sim}|_R$ is bounded reflexive.

Lemma 3.3 Let $A \in N_T(H)$ then A is positive.

Proof; Let $A \in N_T(H)$. By definition A is said to be positive if $\langle Ax, x \rangle \geq 0$ for any $x \in H$. Similarly, every positive operator on a complex Hilbert space is self adjoint from **[Lemma 3.1]** that is $A = A^*$ and $A - A^* \geq 0$ and $\langle Ax, x \rangle \leq \langle A^* x, x \rangle$ and for x is positive. If A is positive then $\langle Ax, x \rangle$ is real. Indeed for $x, y \in H$ then we can have $\langle A(x,y), x+y \rangle \leq \left[\langle Ax, x \rangle^{\frac{1}{2}} + \langle Ay, y \rangle^{\frac{1}{2}} \right]^2$ such that $\langle A(x,y), x+y \rangle = \langle Ax, x \rangle + 2\operatorname{Re}\langle Ax, y \rangle + \langle Ay, y \rangle$

and by the Cauchy-Bunyakovsky-Schwarz inequality $2\mathrm{Re}\langle Ax, y\rangle \leq [\langle Ax, x\rangle\langle Ay, y\rangle]^{\frac{1}{2}}$. Therefore, $\langle y, Ax\rangle = \langle A^{*}y, x\rangle$ such that $AA^{*} \geq 0$, then $\langle x, Ay\rangle = \langle Ax, x\rangle = \|Ax\| \geq 0$. Hence $A \in N_T(H)$ is a positive operator.

Theorem 3.4 Let A, B be in $N_T(H)$ and $\lambda \in C$ then $A+B, A-B$ and λA are in $N_T(H)$.

Proof; Let $A+B$ and λA be bounded linear operators for all scalars λ and $\|A+B\| \leq \|A\| + \|B\|$, $\|\lambda A\| = |\lambda|\|A\|$, Moreover, $\|A\| = 0$ if and only if $A = 0$ .Thus the vector space $S(N_T(H))$ of bounded linear operators from A to B on a complex Hilbert space. Therefore, $\|(A+B)x\| \leq \|Ax\| + \|Bx\| \leq (\|A\| + \|B\|)\|x\|$ for all $x \in H$, hence $A+B$ is bounded, and $\|A+B\| \leq \|A\| + \|B\|$. Thus $A+B$ is in $N_T(H)$. From the fact that $\|(\lambda A)\| = |\lambda|\|A\|$ we can have $\|(\lambda A)x\| = |\lambda|\|Ax\|$ for all $x \in H$ and therefore λA is bounded [see the [**Theorem 3.2**].

Proof; $A-B$, Since $A, B \in N_T(H)$ then we have $\|A\| - \|B\| = \|(A-B)+B\| - \|B\| \leq (\|A-B\| + \|B\|) - \|B\| = \|A-B\|$, and $\|B\| - \|A\| \leq \|A-B\|$, therefore, we have $|\|A\| - \|B\|| \leq \|A-B\|$.

Theorem 3.5 Let A be in $N_T(H)$ then A is contractive if it is the identity.

Proof: Let H be a complex Hilbert space. Recall that a bounded linear operator $A \in N_T(H)$ is contractive if $\|A\| \leq 1$ for instance $\|Ax\| \leq \|x\|$ for any $x \in H$. Therefore, let $A \in N_T(H)$ and I to be identity operator and $A^{*} \in N_T(H)$ be the adjoint of $A \in N_T(H)$ in a bounded linear operators in a Hilbert spaces. Similarly, recall that $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^{*}Ax, x\rangle$ and $\|x\|^2 = \langle x, x\rangle$. Then $\|Ax\| \leq \|x\|$ if and only if $\langle A^{*}Ax, x\rangle \leq \langle x, x\rangle$ is similarly to $\|Ax\| \leq \|I\|$. Hence A is contractive if it is the identity.

4. Conclusion

In conclusion we have investigated certain properties of the class of (α, β)-normal transaloid operators. We have shown that (α, β)-normal transaloid operators are bounded, continuous and self adjoint.

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