

NORMS AND NUMERICAL RADII
INEQUALITIES FOR (α, β) -NORMAL
TRANSALOID OPERATORS

BY

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A Thesis Submitted in Partial Fulfilment of the Requirements for the
Award of the Degree of Master of Science in Pure Mathematics

School of Mathematics and Actuarial Science

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AND TECHNOLOGY


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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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ACKNOWLEDGMENTS

Above all, I thank the Almighty God for His faithfulness and love for me. My sincere gratitude goes to Dr. Benard Okelo and Prof. Omolo Ongati who taught me all through my undergraduate studies and also at masters level. I am extremely grateful to Dr. Benard Okelo for his patience, perseverance, understanding and readiness for consultations which has made this study to reach this far. I am also indebted to JOOUST for offering me the opportunity to study in this esteemed institution. To my classmates Moses Ochola and Kangogo Willy who kept encouraging me and ready to help me at any time I say thank you. Finally, I thank my mother for her prayers, encouragements and provisions towards my success in completing this thesis.

DEDICATION

*To my beloved wife, Sarah, my daughters, Emmaculate and Juliet and
my mother, Kiemisi.*

ABSTRACT

Studies have been carried out on Hilbert space operators especially on operator inequalities related to operator norms, numerical ranges and numerical radii for a family of bounded linear operators acting on a Hilbert space and other non-normal operators like normaloid operators, transaloid operators, self-adjoint operators and normal operators among others. The norm and numerical radii inequalities have been established by many mathematicians for example Dragomir, Kittaneh, Moslehian, Sevilla and Malaga among others. Recently, results have been obtained by Dragomir and Moslehian on norms and numerical radii for (α, β) - normal operators but not for normal transaloid operators. Therefore, in this study we investigated properties of (α, β) - normal transaloid operators and determined their norms. The study has been guided by the following objectives: To investigate (α, β) - normal transaloid operators, to determine norm inequalities for (α, β) - normal transaloid operators and to determine numerical radii inequalities for (α, β) - normal transaloid operators. The methodology involved the use of known inequalities like Cauchy-Schwarz inequality, Triangle inequality and parallelogram law. We also used the approach of direct sum and numerical ranges. The results obtained show that (α, β) -normal transaloid operators satisfy linearity, continuity, positivity, boundedness, reflexivity and self-adjointness. Norm inequalities and numerical radii inequalities have also been obtained. These results may be useful in applications in quantum theory, particularly in commutator approximation and estimation of ground state energies.

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Chapter 1

INTRODUCTION

1.1 Mathematical background

Studies on the properties of Hilbert space operators which include spectrum, numerical ranges, numerical radii and norms are fundamental in various fields of mathematics including operator theory, trigonometry, numerical analysis, fluid dynamics among others. Operator inequalities related to operator norms and numerical radii for a family of bounded linear operators acting on a Hilbert space have also been studied for instance; Dragomir and Moslehian [5] among others. These studies can also be found in [1, 4, 6, 8, 12, 27, 29].

Most mathematicians and researchers have devoted themselves in various fields of linear operator theory in Hilbert spaces, for instance, mathematical inequalities and structural properties of linear operators and this has grown over the last decade. Melbourne [31] studied numerical radii and norm inequalities for some operators on complex Hilbert spaces. The techniques employed by [31] in proving the result are elementary with some special vector inequalities in inner product spaces due to Buzano,

Goldstein, Ryff and Clerk as well as some reverse Schwarz inequalities due to Dragomir [10, 11, 14, 15, 16].

Kittaneh [22, 23, 24, 25] did extensive work on norm inequalities and numerical radius inequalities and established excellent results, for instance new numerical radius inequalities involving products and commutators of positive operators which are essential for diverse applications to other classes of non-normal operators. Khalid and Hussien [26] studied general numerical radius inequality and general norm inequality for Hilbert space operators. Wathiq [37] recently, obtained results on generalized numerical radius inequalities for Operator matrices. Moreover, Wathiq and Kittaneh [38] established new norm equalities and inequalities for operator matrices by comparing norms of $A = [A_{jk}]$ and those of A_{jk} , $j, k = 1, 2, \dots, n$.

Takaki and Masahiro [39] characterized log-hyponormal operators via p -paranormality and brought out clear inclusion relations among other classes of operators but not on transaloid operators. Carlos [2] provides basic knowledge in understanding the properties and characterizations of classes of operators in Hilbert spaces. However, the results obtained by [2] are not on transaloid operators. Jiankui and Zhidong [21] studied on bounded reflexivity of operator spaces in which the connection between the study of complete positivity of elementary operators and reflexivity of convex sets of operators is well defined. Similarly, Zhidong [40] established excellent results of commutants and hyporeflexive closure of operators and Janko [20] studied on reflexivity of commutant and local commutants of an algebraic operator. However, in their research work they did not study on transaloid operators. Christopher [3] and Sylvia [36] highlighted essential properties of operators on Hilbert spaces which added some ingredients

to our study. From the research done by several authors, it is clear that much has been done on numerical radii and numerical ranges in Hilbert space operators, for instance in [1, 9, 11, 19, 28, 33], but little has been done on norms and numerical radii inequalities for (α, β) - normal transaloid operators in Hilbert spaces. Fujii and Nakamoto [17] characterized transaloid operators in terms of spectral sets and dilations and other non-normal operators for instance normaloid operators. Mohammed, Ali and Issam [30] studied on numerical radius inequalities for the Cartesian decomposition of bounded linear operators in Hilbert Spaces and obtained results which are applicable to many classes of operator theory. Recently, Dragomir and Moslehian [4] studied norms and numerical radii for (α, β) -normal operators. Also, Senthilkumar[34] studied on $p - (\alpha, \beta)$ -normal operators. In this study, we considered norm inequalities and numerical radii inequalities for (α, β) - normal transaloid operators. Subsequently, we reviewed some basic concepts and definitions which were useful to our work as outlined in the section below.

1.2 Basic concepts

Definition 1.1. A Field is a set \mathbb{K} together with two binary operations $(+)$ and (\cdot) for which the following conditions hold:

- (i). (Closure) for all $u, v \in \mathbb{K}$ the sum $u + v$ and product $u \cdot v$ belongs to \mathbb{K} .
- (ii). (Associativity) for all $u, v, w \in \mathbb{K}$, $(u + v) + w = u + (v + w)$ and $u \cdot (v \cdot w) = (u \cdot v) \cdot w$.

- (iii). (Commutativity) for all $u, v \in \mathbb{K}$ and $u.v = v.u$.
- (iv). (Distributive Laws) for all $u, v, w \in \mathbb{K}$, $u.(v + w) = u.v + u.w$.
- (v). (Existence of an additive identity) For all $u \in \mathbb{K}$, $\exists 0 \in \mathbb{K}$ such that $u + 0 = u$ and $0 + u = u$.
- (vi). (Existence of multiplicative identity) $\exists 1 \in \mathbb{K}$ with $1 \neq 0$, $u.1 = u$ and $1.u = u$ for all $u \in \mathbb{K}$.
- (vii). (Existence of additive inverse) for all $u \in \mathbb{K}$, $\exists y \in \mathbb{K}$, $u + y = 0$ and $y + u = 0$
if $y = -u$ is the additive inverse of \mathbb{K} .
- (viii). (Existence of multiplicative inverse) for all $y \in \mathbb{K}$ with $y \neq 0$, then $u.y = 1$ and $y.u = 1$ where $u = y^{-1}$.

Definition 1.2. A Vector Space over the field \mathbb{K} is a set V on which two operations called addition and scalar multiplication denoted by $(+)$ and (\cdot) are defined and satisfy the following properties:

- (i). (Closure) for all $s \in \mathbb{K}$ and $u, v \in V$, $v + u$ and $s.u$ are uniquely defined in V .
- (ii). (Associativity) for all $w, u, v \in V$ and $s, y \in \mathbb{K}$, $u + (v + w) = (u + v) + w$ and $y(s.u) = (y.s)u$.
- (iii). (Commutativity of addition) for all $u, v \in V$, $u + v = v + u$.
- (iv). (Distributivity laws) for all $y, s \in \mathbb{K}$ and all $u, v \in V$, $y.(u + v) = y.u + y.v$ and $s(u + v) = s.u + s.v$.
- (v). (Unitary law) for all $u \in V$, $1.u = u$.

(vi). (Existence of additive identity) for all $0 \in V$ for which $u + 0 = u$ and $0 + u = u$ for all $u \in V$.

(vii). (Existence of additive inverse) let $u \in V \exists b \in V$ such that $b + u = 0$ and $u + b = 0$, $b = -u$ denoted by $-u$.

Definition 1.3. Let X be a linear space. A non-negative real valued function $\|\cdot\| : X \rightarrow \mathbb{R}$ is called a **norm** on X if it satisfies the following properties for all $x, y \in X$:

(i). $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ (positivity).

(ii). $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{C}$ (Homogeneity).

(iii). $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is a normed space.

Definition 1.4. The numerical range of an operator $T \in B(H)$ is the subset of the complex numbers, \mathbb{C} , defined by $W(T) = \{\langle Tx, x \rangle, x \in H, \|x\| = 1\}$. The numerical radius is given by $w(T) = \sup\{|\langle Tx, x \rangle|, x \in H, \|x\| = 1\}$.

Definition 1.5. Let $B(H)$ be the algebra of all bounded linear operators on H . The spectrum of $T \in B(H)$ denoted by $\sigma(T)$ is defined as $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$. The spectral radius is given by $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$.

Definition 1.6. A Banach space is a complete normed space.

Definition 1.7. Let X be a vector space over \mathbb{K} . A non-negative real valued function $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$ is called an inner product space if for $x, y \in X$ and $\alpha \in \mathbb{K}$ satisfies the following properties:

(i). $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(ii). $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(iii). $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry).

The pair $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Definition 1.8. A Hilbert Space is a complete inner product space.

Definition 1.9. Let $T : H \rightarrow H$ be a linear operator. Then T is said to be a normaloid operator if $\|T\| = w(T)$.

Definition 1.10. Let $T : H \rightarrow H$ be a linear operator. Then T is said to be a spectraloid operator if $w(T) = r(T)$.

Definition 1.11. Let V be a normed space. A set S is said to be convex if given any two points $a, b \in S$ in a line segment then $\{\lambda a + (1 - \lambda)b : \lambda \in [0, 1]\}$ is also contained in S .

Definition 1.12. The convex hull of a set S of points is defined by $ConvS = \{ \sum_i \lambda_i P_i | \lambda_i = 1, \geq 0 \}$.

Definition 1.13. Let $B(H)$ be the Banach algebra of all bounded linear operators and $T \in B(H)$, then T is said to be convexoid if $\overline{W(T)} = Conv\sigma(T)$ where $Conv\sigma(T)$ is the convex hull of the spectrum T in the complex plane.

Definition 1.14. Let $T : H \rightarrow H$ be a linear operator. Then T is said to be normal if $TT^* = T^*T$.

Definition 1.15. A linear operator $T : H \rightarrow H$ is called self adjoint if $T = T^*$.

Definition 1.16. Let $T \in B(H, S)$ where H and S are Hilbert spaces, then the linear operator $T^* \in B(S, H)$ satisfies $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in H$ and $y \in S$ is called the adjoint operator of T .

Definition 1.17. An operator T is said to be bounded if for $T : H \rightarrow H$ there exist a constant $M \geq 0$ such that: $\|Tx\| \leq M\|x\| \forall x \in H$.

Definition 1.18. An operator T is said to be normaloid if $\|T\| = w(T)$ and transaloid if $\lambda \in \mathbb{C}$ such that $T_\lambda = T - \lambda I$.

Definition 1.19. An operator T acting on a complex Hilbert space H is called (α, β) -normal operator ($0 \leq \alpha \leq \beta$) if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$.

Definition 1.20. Let $T \in B(H)$. T is said to be a (α, β) -normal transaloid operator if the following conditions hold:

(i). $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T \quad \forall 0 \leq \alpha \leq \beta$.

(ii). $T - \lambda I \quad \forall \lambda \in \mathbb{K}$ is normaloid.

Definition 1.21. Let S be a subspace of $B(X)$ and let $ref(S) = \{T \in B(X) : Tx \in [Sx], x \in X\}$ where $[.]$ denotes the norm closure then $S \subseteq B(X)$ is said to be reflexive if $ref(S) = S$.

Definition 1.22. Given a matrix $A = (a_{ij}) \in Mn(\mathbb{C})$, then trace $tr(A)$ of A is the sum of diagonal elements that is $tr(A) = a_{11} + a_{22} + \dots + a_{nn}$. Trace is a linear map such that $tr(\gamma A) = \gamma tr(A)$ and $tr(A+B) = tr(A) + tr(B)$.

Remark 1.23. The class of all (α, β) -normal transaloid operators on H is denoted by $N_T(H)$.

1.3 Statement of the problem

Many researchers and mathematicians have done extensive work on various inequalities for normal operators in Hilbert spaces. Also, a lot has been done on numerical ranges, numerical radii of convexoid, normaloid, spectraloid, and other classes of non-normal operators in operator theory. Dragomir and Moslehian recently worked on various inequalities between norm and numerical radius of (α, β) -normal operators. Senthilkumar studied generalization of p - (α, β) -normal operators but not for (α, β) -normal transaloid operators. Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H . In this study we investigate the properties of (α, β) -normal transaloid operators, determine norm inequalities for (α, β) -normal transaloid operators. Determine numerical radii inequalities for (α, β) -normal transaloid operators.

1.4 Objectives of the study

This study has been guided by the following objectives:

- (i). To investigate the properties of (α, β) -normal transaloid operators.
- (ii). To determine norm inequalities for (α, β) -normal transaloid operators.
- (iii). To determine numerical radii inequalities for (α, β) -normal transaloid operators.

1.5 Significance of the study

A lot has been done on numerical ranges, numerical radii inequalities of bounded linear operators in Hilbert spaces and normal operators by many mathematicians for example Dragomir, Kittaneh, Moslehian among others. But the results on normal transaloid operators have not been fully established. The results we have obtained may have great importance in various fields such as quantum mechanics and a contribution of knowledge to operator theory.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we review related literature. We considered norm inequalities, numerical radii inequalities and general theory of (α, β) -normal operators and transaloid operators.

2.2 Norm inequalities

Dragomir and Moslehian [4] established various inequalities between the operator norm and numerical radius of (α, β) -normal operators in Hilbert spaces which are significant to our work. These results are:

Theorem 2.1. [4, Theorem 2.1] *Let $T \in B(H)$ be an (α, β) -normal operator. Then $(\alpha^{2r} + \beta^{2r})\|T\|^2 \leq \{2\beta^2 w(T^2) + r^2 \beta^{2r-2} \|\beta T - T^*\|^2, r \geq 1$
 $2\beta^2 w(T^2) + \|\beta T - T^*\|^2, r < 1\}$.*

Theorem 2.2. [4, Theorem 2.2] *Let $T \in B(H)$ be an (α, β) -normal operator. Then $w(T)^2 \leq \frac{1}{2}[\beta\|T\|^2 + w(T^2)]$.*

From Theorem 2.1 and Theorem 2.2 the researchers focused on inequalities involving norm and numerical radius of (α, β) - normal operators but not for normal transaloid operators. However, Dragomir [11] further worked out on some inequalities for normal operators in Hilbert spaces by use of classical inequalities for vectors in inner product spaces as shown below.

Theorem 2.3. [11, Theorem 4] Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \rightarrow H$ a normal linear operator on H then $\frac{1}{2}(\|Tx\|^2 + |\langle T^2x, x \rangle|) \geq |Tx, x|^2$ for any $x \in H, \|x\| = 1$.

Theorem 2.4. [11, Theorem 10] Let $T : H \rightarrow H$ be a normal operator on the Hilbert space if $\lambda \in \mathbb{C}, |\lambda| \neq 1$ then $\|T - |\lambda|^{v+1}T^*\| \leq \frac{1-|\lambda|^{v+1}}{1-|\lambda|} \|T - \lambda T^*\|$, for any $v > 0$.

Theorem 2.5. [11, Theorem 13] Let $T : H \rightarrow H$ be a normal operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $\alpha, \beta \in \mathbb{C}$. Then $\|T\|^p [(|\alpha + \beta|)^p + \|\alpha\| - \|\beta\|]^p \leq \|\alpha T + \beta T^*\| + \|\alpha T - \beta T^*\|^p$ if $p \in (1, 2)$ and $\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \geq 2(|\alpha|^p + |\beta|^p)\|T\|^p$.

Theorem 2.6. [10, Theorem 14] Let $T : H \rightarrow H$ be normal operator on the Hilbert space. If $\alpha, \beta \in \mathbb{C}$ and r, γ such that $\|T - \alpha I\| \leq r$ and $\|T^* - \beta I\| \leq \gamma$, then $\|T\|^2 + \frac{1}{2}(|\alpha|^2 + |\beta|^2) \leq \frac{1}{2}(r^2 + \gamma^2) + \|\alpha T - \beta T^*\|$.

Theorem 2.7. [5, Theorem 5] Let $T : H \rightarrow H$ be a normal operator on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If $\lambda \in \mathbb{C}$ then $\|Tx\|^2 - |\langle T^2x, x \rangle| \leq \frac{2}{(1+|\lambda|)^2} \|Tx - \lambda T^*\|^2$ for any $x \in H, \|x\| = 1$.

From Theorem 2.3, Theorem 2.4, Theorem 2.6 and Theorem 2.7, Dragomir researched extensively on some inequalities for normal operators in Hilbert spaces and numerical radius inequalities but not for normal transaloid operators. In this study we employed known inequalities for example, Cauchy-Schwarz inequality in determining the norms for normal transaloid operators. Fujii and Nakamoto [17] explained transaloid operator in terms of spectral sets and dilations and other non-normal operators. For instance in characterization of normaloids they gave the following result:

Theorem 2.8. [17, Theorem 1] *An operator T is normaloid if and only if $r(T)D$ is a spectral set for T .*

Theorem 2.8 justifies that T is normaloid if and only if $\|T\| = w(T)$. On characterization of transaloid operators the following two theorems were given:

Theorem 2.9. [17, Theorem 3] *An operator T is transaloid if and only if any disk containing the spectrum $\sigma(T)$ is a spectral set for T .*

Theorem 2.10. [17, Theorem 4] *An operator T is transaloid if and only if any disk containing $\overline{W(T)}$ is a spectral set for T .*

Theorem 2.9 is a consequence of theorem 2.10 by the fact that a disk contains $\sigma(T)$ if and only if it contains $W(T)$. An operator T is called numeroid if $\overline{W(T)}$ a spectral set for T .

Theorem 2.11. [17, Theorem 5] *A numeroid is transaloid.*

From Theorem 2.8, Theorem 2.9, Theorem 2.10 and Theorem 2.11 Fujii and Nakamoto [17] characterized transaloid operators in terms of spectral sets, dilations and other non-normal operators but not (α, β) -normal

transaloid operators and our study is to investigate the properties of (α, β) - normal transaloid operators in Hilbert spaces, using other techniques other than spectral sets. Takayuki Furuta [18] studied a lot on various classes of non-normal operators on Hilbert spaces and indeed on characterization of the normaloid operators as seen in the result below:

Theorem 2.12. [18, Theorem 2] *The following assertions are mutually equivalent:*

- (i). *T is a normaloid operator if $\|T\| = r(T)$.*
- (ii). *$\|T^n\| = \|T\|^n$ for all natural numbers.*
- (iii). *$\|T\| = w(T)$.*

On normal and self adjoint operators, Dragomir and Sandor [7], Sevilla and Malaga [35] have intensively worked on various classes of Hilbert spaces for instance on normal operators, self-adjoint operators, adjoint operators and among others.

Proposition 2.13. [35, Proposition 2.2] *Let $T : H_1 \rightarrow H_2$ be a bounded operator between two Hilbert spaces. There exist a unique operator $T^* : H_2 \rightarrow H_1$ called the adjoint operator of T such that $\langle Tx, y \rangle_{H_2} = \langle x, T^*y \rangle_{H_1}$ $\forall x \in H_1, y \in H_2$.*

Proposition 2.14. [35, Proposition 2.4] *An operator $T \in B(H)$ is normal if and only if $\|Tx\| = \|T^*x\| \forall x \in H$.*

Theorem 2.15. [7, Theorem 1.3] *A necessary and sufficient condition for an operator T to be normal is that $\|Tx\| = \|T^*x\|$ for every vector $x \in H$. We can observe that, if one uses Cauchy-Schwarz inequality we*

can have $|\langle u, v \rangle| \leq \|u\| \|v\|$ for all $u, v \in H$. For the choices $u = Tx$, $v = T^*x$ then one gets the following simple inequality for normal operator T ; $|\langle Tx, T^*x \rangle| \leq \|Tx\| \|T^*x\| \|Tx\|^2 \geq |T^2x, x|, \forall x \in H$.

Proposition 2.13, proposition 2.14 and Theorem 2.15 they provides necessary and sufficient conditions for normality of operators, but not for normal transaloid operators. In our study, we are using the Cauchy-Schwarz inequality as well as triangle inequality in estimating norms of transaloid operators.

Khalid and Hussien [26] determined general numerical radius inequality for Hilbert space operators which yields well known and new numerical radius inequalities as shown in the following basic theorem.

Theorem 2.16. *Let $A_i, B_i, X_i \in B(H)$ ($i = 1, 2, \dots, n$), and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t \forall t \in [0, \infty)$. Then,*

$w^r(\sum_{i=1}^n A_i^ X_i B_i) \leq \frac{n^{r-1}}{2} \| \sum_{i=1}^n ([A_i^* g^2(|X_i|) A_i]^r + [A_i^* f^2(|X_i|) B_i]^r) \|$ for all $r \geq 1$.*

Jiankui and Zhidong [21] researched on bounded reflexivity of operator spaces on finite dimensional subspaces of $B(X)$ with separating vector and established the following:

Corollary 2.17. *[21, Corollary 3.6] If S is a finite dimensional subspace of $B(X)$ and S has a separating vector, then S is boundedly reflexive. In particular, if $\dim S = n$ and every non-zero operator in S has rank greater than or equal to n , then S is boundedly reflexive.*

Corollary 2.18. [21, Corollary 4.8] Let S and \tilde{S} be subspaces of $B(X)$ with a separating vector space x . If $\dim(\tilde{S}|_m)$ is finite, then S is algebraically boundedly reflexive.

Jiankui and Zhidong [21] studied only on bounded reflexivity of general operators but not for the class of (α, β) -normal transaloid operators. Also from the research done by Khalid and Hussien [26] it was purely on determination of general numerical radius inequality for Hilbert spaces but not for (α, β) -normal transaloid operators. However, we require the application of their results to our work on determining the norm inequalities for the class of (α, β) -normal transaloid operators.

2.3 Numerical radii inequalities

On some numerical radius inequalities for Hilbert space operators Moslehian and Assadollah [32] established general numerical inequalities as from the following corollaries and theorems:

Corollary 2.19. [32, Corollary 2.6] If $A \in B(H)$, then $w(A) = \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \leq \|A\|$.

Corollary 2.20. [32, Corollary 2.7] If $A \in B(H)$, then $\|A + A^*\| \leq \|A\| + \|A^2\|^{\frac{1}{2}} \leq 2\|A\|$.

Theorem 2.21. [32, Theorem 2.9] Suppose that $A \in B(H)$ is positive. Then $w(AX - XA) \leq \frac{1}{2}\|A\|(\|X\| + \|X^2\|^{\frac{1}{2}})$.

Fuad Kittaneh [24] also worked out on numerical radius inequality by use

of mixed Schwarz inequality and arithmetic-geometric mean inequality as indicated by the following Lemmas and theorems:

Lemma 2.22. [24, Lemma 1] If $A \in B(H)$, then $\|\langle Ax, y \rangle\| \leq \langle |A|x, x \rangle^{\frac{1}{2}} \langle |A^*|y, y \rangle^{\frac{1}{2}}$
 $\forall x, y \in H$.

Lemma 2.23. [24, Lemma 2] If $A, B \in B(H)$ are positive operators, then $\|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \|AB\|^{\frac{1}{2}}$.

Theorem 2.24. [24, Theorem 1] If $A \in B(H)$, then $w(A) = \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}})$.

From the above Corollary 2.19 and Corollary 2.20, Lemma 2.22, Lemma 2.23, Theorem 2.24 and Theorem 2.21, Moslehian and Assadollah [32] and Fuad Kittaneh [24] used the mixed Schwarz inequality and arithmetic-geometrical mean inequality to obtain numerical radius inequalities for the normal operators but not on normal transaloid operators, similarly, we require the Cauchy-Schwarz inequality and other inequalities like parallelogram law and triangle inequality to determine numerical radii inequalities for (α, β) - normal transaloid operators. Actually, the study of the numerical radii inequalities has been of great interest to mathematicians in recent times. Bonyo, Adicka and Agure [1] also contributed on generalization of numerical radius inequalities as shown in the theorems below;

Theorem 2.25. [1, Theorem 2.5] Let A and B be self-adjoint operators in $B(H)$ and $r \geq 1$. Then $w^r(A + B) \leq 2^{r-1}(\|A\|^r + \|B\|^r)$.

Theorem 2.26. [1, Theorem 2.10] Let A and B be self-adjoint operators in $B(H)$ and $r \geq 2$. Then $w^r(A + B) \leq 2^{r-2}(\|A + B\|^r + \|A - B\|^r)$.

Also Omid, Alireza and Mohsen [33] obtained results on numerical radius inequalities for finite sums of operators and determined an upper bound for numerical radius by means of the cartesian decomposition of operators as shown from the following theorems:

Theorem 2.27. [33, Theorem 2.5] Let $X, Y \in B(H)$ and $r \geq 1$, $r' \geq 1$ and $0 \leq \alpha \leq 1$. Then for each unit vector $x \in H$, $|\langle Xx, x \rangle|^r |\langle Yx, x \rangle|^{r'} \leq \frac{1}{2}(\|\alpha|X|^{2r} + (1-\alpha)|X^*|^{2r} + \alpha|Y|^{2r'} + (1-\alpha)|Y^*|^{2r'}\|)$.

Theorem 2.28. [33, Theorem 3.1] Let $A_j \in B(H)$ have the cartesian decomposition $A_j = B_j + C_j$ for all $j = 1, \dots, n$ and $r \geq 1$. Then $w^r(\sum_{j=1}^n A_j) \leq (\sqrt{2n})^{r-1} \sum_{j=1}^n (\|B_j\|^{2r} + \|C_j\|^{2r})^{\frac{1}{2}}$.

From the above literature review, it is evident that a lot has been done on normal, self adjoint and (α, β) - normal operators but not on the class of (α, β) - normal transaloid operators. Therefore, we have investigated properties of (α, β) - normal transaloid operators, determined the norm inequalities for (α, β) - normal transaloid operators and determined numerical radii inequalities for (α, β) - normal transaloid operators.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

For a successful completion of this research, background knowledge on numerical range, numerical radius of all bounded linear operators in Hilbert spaces, inequalities for normal operators in Hilbert spaces (especially classical inequalities, operator theory, non-normal operators and functional analysis) is very crucial. We also stated some known inequalities and fundamental results but omitted the proofs shown in the sections hereafter.

3.2 Fundamental results and theorems

We have stated some known results which are fundamental to our work but we omit the proofs.

Theorem 3.1. [11, Theorem 13] Let $T : H \rightarrow H$ be a normal operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $\alpha, \beta \in \mathbb{C}$. Then $\|T\|^p(|\alpha + \beta|^p + \|\alpha - \beta\|^p) \leq \|\alpha T + \beta T^*\| + \|\alpha T - \beta T^*\|^p$ if $p \in (1, 2)$ and $\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \geq 2(|\alpha|^p + |\beta|^p)\|T\|^p$ if $p \geq 2$.

Proof. See [11, Theorem 13]. □

Theorem 3.2. [4, Theorem 3.1] Let $T \in B(H)$ be an (α, β) -normal operator. If $p \geq 2$, then $(1 + \alpha^q)\|T\|^p \leq \frac{1}{4}(\|T + T^*\|^p + \|T - T^*\|^p)$.

Proof. See [4, Theorem 3.1]. □

Theorem 3.3. [22, Theorem 1] If $A \in B(H)$, then $\frac{1}{4}(\|A^*A + AA^*\| \leq w((A))^2 \leq \frac{1}{2}\|A^*A + AA^*\|$.

Proof. See [22, Theorem 1]. □

Theorem 3.4. [13, Theorem 3.4] If $A_1, \dots, A_n \in B(H)$, then $\|\sum_{k=1}^n A_k\|^2 \leq 4w(\sum_{k=1}^n A_k) \sum_{j=1}^n w(A_j)$. Moreover, if $\sum_{k=1}^n A_k$ commutes with each A_j^* for $j \in \{1, \dots, n\}$, then $\|\sum_{k=1}^n A_k\|^2 \leq 2w(\sum_{k=1}^n A_k) \sum_{j=1}^n w(A_j)$.

Proof. See [13, Theorem 3.4]. □

Theorem 3.5. [6, Theorem 4.3] Let $A, B : H \rightarrow H$ be two bounded linear operators on the Hilbert space H . Then $\|\frac{A+B}{2}\|^2 \leq \frac{1}{2}\|\frac{A^*A+B^*B}{2}\| + w(B^*A)$.

Proof. See [6, Theorem 4.3]. □

3.3 Known inequalities

The methodology involves the use of Cauchy-Schwarz inequality, Parallelogram law, and Triangle inequality. We also used the approach of polar decomposition and direct sum.

Definition 3.6. Let V be an inner product space. Then for all $x, y \in V$ we have $|\langle x, y \rangle| \leq \|x\| \|y\|$. This is called Cauchy-Schwarz Inequality.

Definition 3.7. Let $x, y \in V$, where V is a normed space. Then $\|x+y\| \leq \|x\| + \|y\|$ is called Triangle inequality.

Definition 3.8. Let $x, y \in V$, where V is an inner product space. The parallelogram law states that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Definition 3.9. Let T be an operator on a Hilbert space H . When $T = U|T|$ with $N(U) = N(|T|)$, $T = U|T|$ is said to be the polar decomposition of T , and if the kernel condition $N(U) = N(|T|)$ is not necessarily satisfied, $T = U|T|$ is said to be a decomposition of T .

Definition 3.10. Let V be vector space and U, W be subspaces of V . Then V is said to be **direct sum decomposition** of subspaces of U_1, \dots, U_k , if it can be expressed as $V = U_1 \oplus \dots \oplus U_k$ for all $v \in V$. Moreover, there exist unique vectors $u_i \in U_i$ for all $1 \leq i \leq k$ such that $v = u_1 + u_2 + \dots + u_k$.

Chapter 4

INEQUALITIES FOR (α, β) - NORMAL TRANSALOID OPERATORS

4.1 Introduction

In this chapter, we investigated properties of (α, β) -normal transaloid operators. We considered the properties of this class of operators and give results on their characteristics. The properties which have been considered include: self-adjointness, linearity, continuity, boundedness, reflexivity and positivity. We have determined the numerical radii inequalities and norm inequalities for (α, β) -normal transaloid operators. Throughout this chapter we denote the class of (α, β) -normal transaloid operators by $N_T(H)$.

Lemma 4.1. *Let $A \in N_T(H)$ then A is self adjoint.*

Proof. We need to show that $A = A^*$. Indeed for any $z \in H$, we can have

$$\overline{\langle Az, z \rangle} = \langle z, Az \rangle = \langle A^*z, z \rangle = \langle Az, z \rangle$$

Therefore, $\langle Az, z \rangle$ is real. Also assume that $\langle Az, z \rangle$ is real for all z . Then choose any $z, g \in H$. We can have $\langle A(z+g), z+g \rangle = \langle Az, z \rangle + \langle Az, g \rangle + \langle Ag, z \rangle + \langle Ag, g \rangle$. It implies that $\langle A(z+g), z+g \rangle, \langle Az, z \rangle, \langle Ag, g \rangle$ are real, We can conclude that $\langle Az, g \rangle + \langle Ag, z \rangle$ is real and it equals its own complex conjugate, for instance

$$\langle Az, g \rangle + \langle Ag, z \rangle = \overline{\langle Az, g \rangle + \langle Ag, z \rangle} = \langle g, Az \rangle + \langle z, Ag \rangle \quad (4.1.1)$$

We can see that

$$i\langle Az, g \rangle + i\langle Ag, z \rangle = \overline{i\langle Az, g \rangle + i\langle Ag, z \rangle} = i\langle g, Az \rangle - i\langle z, Ag \rangle$$

by multiplying i throughout, we have

$$i(i\langle g, Az \rangle - i\langle z, Ag \rangle) = -\langle g, Az \rangle + \langle z, Ag \rangle. \quad (4.1.2)$$

By adding 4.1.1 and 4.1.2 we obtain $2\langle Az, g \rangle = 2\langle z, Ag \rangle = 2\langle A^*z, g \rangle$. Therefore, this is true for z and g , we can conclude that $A = A^*$. Hence A is self adjoint. \square

Theorem 4.2. Let $A \in N_T(H)$ then A is linear, bounded, continuous and reflexive.

Proof. For the linearity;

Let $A \in N_T(H)$. Let $y_1, y_2 \in H$ and $\alpha, \beta \in \mathbb{C}$, then for $x \in H$ we have

$$\begin{aligned} \langle x, A^*(\alpha y_1 + \beta y_2) \rangle &= \langle Ax, \alpha y_1 + \beta y_2 \rangle \\ &= \bar{\alpha} \langle Ax, y_1 \rangle + \bar{\beta} \langle Ax, y_2 \rangle \\ &= \langle x, \alpha A^* y_1 + \beta A^* y_2 \rangle. \end{aligned}$$

It follows that $A^*(\alpha y_1 + \beta y_2) = \alpha A^* y_1 + \beta A^* y_2$. i.e A is linear.

Also A is bounded. Put $x = A^* y$. Then

$$\begin{aligned} \|A^* y\|^2 &= \langle A^* y, A^* y \rangle \\ &= \langle AA^* y, y \rangle \\ &\leq \|AA^* y\| \|y\| \\ &\leq \|A\| \|A^* y\| \|y\|. \end{aligned}$$

So that $\|A^* y\| \leq \|A\| \|y\|$ for any $y \in H$, i.e $A \in N_T(H)$ is bounded.

Similarly, $A \in N_T(H)$ with a linear operator $A : H_1 \rightarrow H_2$ is continuous if x_n and x are sequences from H , such that $x_n \rightarrow x$, then we have

$$\begin{aligned} \|Ax_n - Ax\| &= \|A(x_n - x)\| \\ &\leq \|A\| \|x_n - x\| \rightarrow 0 \end{aligned}$$

i.e $Ax_n \rightarrow Ax$, which proves the continuity of $A \in N_T(H)$.

For the proof of reflexivity of an operator we need to consider V and \tilde{V} to be finite dimensional subspaces of $N_T(H)$, then $V \cap \dim(\tilde{V}|_R)$ is finite. By [21, Corollary 4.8] it is sufficient enough to show that $(\tilde{V}|_R)$ is algebraically bounded and reflexive since algebraic bounded reflexivity implies bounded reflexivity, hence $(\tilde{V}|_R)$ is bounded reflexive. To see this, let $W \in \text{ref}_b(\tilde{V}|_R)$. Let \tilde{W} be the extension of W to $[R]$, where $[R]$ is the norm closure of R . By $W \in \text{ref}_b(\tilde{V}|_R)$, $\exists R_W$ such that $\forall a \in R$, $W_a \in [V_{R_W} a]$. Then $\dim(\tilde{V}|_R)$ is finite. Since R contains separating vector of \tilde{V} , if V is a finite dimensional subspace of $B(H)$ with a separating vector then V is boundedly reflexive. Moreover, we can have $W \in \tilde{V}|_{[R]}$. Hence $W \in \tilde{V}|_R$ and $\tilde{V}|_R$ is bounded reflexive. \square

Lemma 4.3. *Let $A \in N_T(H)$ then A is positive.*

Proof. Let $A \in N_T(H)$. By definition A is said to be positive if $\langle Ax, x \rangle \geq 0$ for any $x \in H$. Similarly, every positive operator on a complex Hilbert space is self adjoint as seen from Lemma 4.1 that is $A = A^*$ and $A - A^* \geq 0$ and $\langle Ax, x \rangle \leq \langle A^*x, x \rangle$ for x is positive. If A is positive then $\langle Ax, x \rangle$ is real. Indeed for $x, y \in H$ then we can have $\langle A(x, y), x + y \rangle \leq [\langle Ax, x \rangle^{\frac{1}{2}} + \langle Ay, y \rangle^{\frac{1}{2}}]^2$ such that $\langle A(x, y), x + y \rangle = \langle Ax, x \rangle + 2\text{Re}\langle Ax, y \rangle + \langle Ay, y \rangle$ and by the CBS inequality $2\text{Re}\langle Ax, y \rangle \leq [\langle Ax, x \rangle \langle Ay, y \rangle]^{\frac{1}{2}}$. Therefore, $\langle y, Ax \rangle = \langle A^*y, x \rangle$ such that $AA^* \geq 0$, then $\langle x, Ay \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0$. Hence $A \in N_T(H)$ is a positive operator. \square

Theorem 4.4. Let $A, B \in N_T(H)$ and $\eta \in \mathbb{C}$ then ηA , $A + B$ and $A - B$ are in $N_T(H)$.

Proof. Let $\eta A \in N_T(H)$ and $\alpha^2 A^* A \leq AA^* \leq \beta^2 A^* A$, we have

$$\alpha^2((\eta A)^* \eta A) \leq (\eta A)(\eta A)^* \leq \beta^2((\eta A)^* \eta A)$$

$$\alpha^2(\eta^* \eta (A^* A)) \leq (\eta \eta^* (AA^*)) \leq \beta^2(\eta^* \eta (A^* A)). \text{ Hence } \alpha^2 A^* A \leq AA^* \leq \beta^2 A^* A.$$

Also $\eta A - \lambda \in N_T(H)$, for if we consider a unilateral shift $U \in N_T(H)$ then $A = U^* - 2$ but 2 is not in $\sigma(U^*)$ as in [34, Theorem 2.2]. Therefore, $\eta A \in N_T(H)$.

Let $A + B \in N_T(H)$ then we have $\alpha^2((A + B)^*(A + B)) \leq (A + B)(A + B)^* \leq \beta^2((A + B)^*(A + B))$

$$\alpha^2[AA^* + 2A^*B^* + B^*B] \leq AA^* + 2A^*B^* + B^*B \leq \beta^2[AA^* + 2A^*B^* + B^*B].$$

By distributing α^2 and β^2 to each summand we have $\alpha^2 AA^* + 2\alpha^2 A^* B^* + \alpha^2 B^* B \leq AA^* + 2A^* B^* + B^* B \leq \beta^2 AA^* + 2\beta^2 A^* B^* + \beta^2 B^* B$. If we take $B^* = S$, then AS is unitary. Then the proof of $2AS$ follows from the prove of part (1) and [34, Theorem 2.10.] Hence $A + B \in N_T(H)$.

Let $A - B \in N_T(H)$, then we have

$$\alpha^2((A - B)^*(A - B)) \leq (A - B)(A - B)^* \leq \beta^2((A - B)^*(A - B))$$

$$\alpha^2[AA^* - 2A^*B^* + B^*B] \leq AA^* - 2A^*B^* + B^*B \leq \beta^2[AA^* - 2A^*B^* + B^*B].$$

By distributing α^2 and β^2 to each summand we have $\alpha^2 AA^* - 2\alpha^2 A^* B^* + \alpha^2 B^* B \leq AA^* - 2A^* B^* + B^* B \leq \beta^2 AA^* - 2\beta^2 A^* B^* + \beta^2 B^* B$. Letting $B^* = S$, and AS to be unitary, then the proof follows immediately from part (1) and [34, Theorem 2.10.] Hence $A - B \in N_T(H)$. \square

Theorem 4.5. Let $A \in N_T(H)$ then A is contractive if it is the identity.

Proof. Recall that a bounded linear operator $A \in N_T(H)$ is contraction if $\|Ax\| \leq \|x\|$. Let $A \in N_T(H)$ and I to be identity operator and $A^* \in N_T(H)$ be the adjoint of $A \in N_T(H)$. Also $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle$ and $\|x\|^2 = \langle x, x \rangle$. Then $\|Ax\| \leq \|x\|$ if and only if $\langle A^*Ax, x \rangle \leq \langle x, x \rangle$ is similarly to $\|Ax\| \leq \|I\|$. Hence A is contractive if it is the identity. \square

4.2 Numerical radii inequalities

Under this section we determine numerical radii inequalities for operators in $N_T(H)$. We start by the following auxiliary results.

Lemma 4.6. Let $A \in N_T(H)$. If A is self adjoint then

$$w(A^*) = \frac{\|A^*\| + \|(A^*)^2\|^{\frac{1}{2}}}{2} = \|A\|.$$

Proof. Let $Q = \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} w(A^*) &\leq \frac{1}{2}(\|Q\| + \|Q^2\|^{\frac{1}{2}}) \\ &= \frac{1}{2}(\|A^*\| + \|(A^*)^2\|^{\frac{1}{2}}) \\ &= \|A^*\|. \end{aligned}$$

But $A = A^*$ that is A is self-adjoint. Therefore, we obtain

$$w(A^*) = \frac{\|A^*\| + \|(A^*)^2\|^{\frac{1}{2}}}{2} = \|A\|. \quad \square$$

Lemma 4.7. Let $A \in N_T(H)$ be positive then $w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}})$.

Proof. By use of Cauchy-Schwarz inequality for every $x \in H$ we obtain

$$\begin{aligned} |\langle Ax, x \rangle| &\leq \langle |A|x, x \rangle^{\frac{1}{2}} \langle |A^*|x, x \rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2}(\langle |A|x, x \rangle + \langle |A^*|x, x \rangle) \\ &= \frac{1}{2} \langle (|A| + |A^*|)x, x \rangle. \end{aligned}$$

But since A is a positive operator in $N_T(H)$, we have

$$\begin{aligned} w(A) &= \frac{1}{2} \langle (|A| + |A^*|)x, x \rangle \\ &\leq \frac{1}{2} \max\{\| |A| \|, \| |A^*| \| + \| |A|^{\frac{1}{2}} |A^*|^{\frac{1}{2}} \| \} \\ &= \frac{1}{2} (\|A\| + \|A^2\|^{\frac{1}{2}}). \end{aligned}$$

Thus $w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}})$. □

Lemma 4.8. Let $A, B \in N_T(H)$, then $w(AB) \leq 2w(A)w(B)$. If $AB = BA$.

Proof. Let $A, B \in N_T(H)$ and let $r \in \mathbb{N}$. Then,

$$\frac{1}{r} w(\text{diag}(\sum_{i=1}^r A_i, \dots, \sum_{i=1}^r A_i) (\text{diag}(\sum_{i=1}^r B_i, \dots, \sum_{i=1}^r B_i))) \leq w(AB).$$

Then by definition of r -pinching operators see [32] and

$$w(A_1 \oplus \dots \oplus A_r) w(B_1 \oplus \dots \oplus B_r) = \max\{w(A_i)w(B_i) : i = 1, 2, \dots, r\},$$
 we have

$$w(\text{diag}(\sum_{i=1}^r A_i, \dots, \sum_{i=1}^r A_i)) w(\text{diag}(\sum_{i=1}^r B_i, \dots, \sum_{i=1}^r B_i))$$

$$\begin{aligned}
&= w(\sum_{i=1}^r A_i)w(\sum_{i=1}^r B_i) \leq \sum_{i=1}^r w(A_i)w(B_i) \\
&\leq r \max \{w(A_i), w(B_i) : i = 1, 2\} = rw(A)w(B).
\end{aligned}$$

Hence, taking $r = 2$, we obtain $w(AB) \leq 2w(A)w(B)$. \square

Theorem 4.9. Let $A_1, A_2, A_3, A_4 \in N_T(H)$ and $B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$. Then

$$\max(w(A_1), w(A_4)) \leq \frac{1}{2}(\|B\| + \|B^2\|^{\frac{1}{2}}) \text{ and}$$

$$\max(w(A_2A_3)^{\frac{1}{2}}, w(A_3A_4)^{\frac{1}{2}}) \leq \frac{1}{2}(\|B\| + \|B^2\|^{\frac{1}{2}}).$$

Proof. First we consider r -pinching in [32], and Consider $[A_{ij}]$ where $A_{ij} \in N_T(H)$ and $i, j = 1, 2, 3, 4$. We define $C(A) = A_1 \oplus A_2 \oplus A_3 \oplus A_4$ to be the 4-pinching of A . Let $(A_1 \oplus A_2 \oplus A_3 \oplus A_4 = \max(w(A_i) : i = 1, 2, 3, 4)$, for all operators $A_1, A_2, A_3, A_4 \in N_T(H)$ we have, $\max(w(A_1), w(A_4)) = w \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} \right)$. Since A_4 is arbitrary,

$$\max(w(A_1), w(A_4)) = w \left(\begin{bmatrix} A_1 & 0 \\ 0 & -A_4 \end{bmatrix} \right). \text{ Let } U = w \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right)$$

be a unitary operator on $H \oplus H$. Then,

$$\max \begin{bmatrix} A_1 & 0 \\ 0 & -A_4 \end{bmatrix} = TU + UT. \text{ Therefore, by [32, Equation (1-8)] we}$$

have $w(AU \pm U^*A) \leq \frac{1}{2}(\|A\| + \|A^*\| + U^*(\|A\| + \|A^*\|)U) \leq \|A\| + \|A^2\|^{\frac{1}{2}}$,

we obtain $\max(w(A_1), w(A_4)) \leq \frac{\|B\| + \|B^2\|^{\frac{1}{2}}}{2}$.

To prove the inequality, $\max(w(A_2A_3)^{\frac{1}{2}}, (w(A_3A_4)^{\frac{1}{2}})) \leq \frac{1}{2}(\|B\| + \|B^2\|^{\frac{1}{2}})$.

Let $(A_1 \oplus A_2 \oplus A_3 \oplus A_4) = \max(w(A_i) : i = 1, 2, 3, 4)$ for all bounded

operators $A_1, A_2, A_3, A_4 \in N_T(H)$, then

$$\begin{aligned} \max(w(A_2A_3), w(A_3A_4)) &= w\left(\begin{bmatrix} A_2A_3 & 0 \\ 0 & A_3A_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}^2 \\ &\leq 2w\left(\begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}\right)^2. \end{aligned}$$

Taking $A_1 = A_2A_3$, then $w(A_2A_3) \leq 2w(A_3)(A_2)$, but since A_2 is arbitrary we have

$$\max(w(A_2A_3), w(A_3A_4)) \leq 2w\left(\begin{bmatrix} 0 & -A_2 \\ A_3 & 0 \end{bmatrix}\right)^2.$$

We observe that $2\left(\begin{bmatrix} 0 & -A_2 \\ A_3 & 0 \end{bmatrix}\right) = TU - UT$.

So by the following inequality

$$w(AU \pm U^*A) \leq \frac{1}{2}(\|A\| + \|A^*\| + U^*(\|A\| + \|A^*\|)U) \leq \|A\| + \|A^2\|^{\frac{1}{2}},$$

we obtain $\max(w(A_2A_3)^{\frac{1}{2}}, w(A_3A_4)^{\frac{1}{2}}) \leq \frac{1}{2}(\|B\| + \|B^2\|^{\frac{1}{2}})$. \square

Theorem 4.10. Let $A \in N_T(H)$, then $w(A)^2 \leq \frac{1}{2}[\alpha\|A\|^2 + w(A^2)]$.

Proof. First we consider Buzano inequality

$$|\langle a, c \rangle \langle c, d \rangle| \leq \frac{1}{2}(\|a\|\|d\| + |\langle a, d \rangle|)$$

for any $a, c, d \in H$ with $\|c\| = 1$. Also from Lemma 4.6 and Lemma 4.7 it implies that for $x \in H$ with $\|x\| = 1$,

and $c = x$, $a = Ax$, $d = A^*x$ we have.

$$\begin{aligned} |\langle Ax, x \rangle \langle x, A^*x \rangle| &\leq \frac{1}{2}(\|Ax\| \|A^*x\| + |\langle Ax, A^*x \rangle|) \\ &\leq \frac{1}{2}(\alpha \|Ax\|^2 + |\langle A^2x, x \rangle|). \end{aligned}$$

Taking the maximum over $x \in H$ with $\|x\| = 1$, we obtain

$$w(A)^2 \leq \frac{1}{2}[\alpha \|A\|^2 + w(A^2)]. \quad \square$$

Theorem 4.11. *Let A and B be self-adjoint operators in $N_T(H)$, and $q \geq 2$. Then $w^q(A+B) \leq 2^{q-1}(\|A\|^q + \|B\|^q)$.*

Proof. From Jessen's inequality [1, Lemma 2.4] $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ for $q \geq 2$, $x \in H$ we have

$$\begin{aligned} |\langle (A+B)x, x \rangle|^q &= |\langle Ax, x \rangle + \langle Bx, x \rangle|^q \\ &\leq (|\langle Ax, x \rangle| + |\langle Bx, x \rangle|)^q \\ &\leq (\langle |A|x, x \rangle + \langle |B|x, x \rangle)^q \\ &\leq 2^{q-1}(\langle |A|x, x \rangle^q + \langle |B|x, x \rangle^q) \\ &\leq 2^{q-1}(\langle |A|^q x, x \rangle + \langle |B|^q x, x \rangle) \\ &= 2^{q-1}(\langle |A|^q + |B|^q x, x \rangle). \end{aligned}$$

Taking the maximum over $x \in H$, we obtain $w^q(A+B) \leq 2^{q-1}(\|A\|^q + \|B\|^q)$. □

Corollary 4.12. *Letting $q = 1$ in Theorem 4.11, we have*

$$w(A+B) \leq \|A\| + \|B\|.$$

Proof. It is clear from Theorem 4.11 that if $q = 1$ we obtain $|\langle (A+B)x, x \rangle| \leq \|A\| + \|B\|$.

$B)x, x| = (\|A\| + \|B\|)|x, x|$. Therefore, taking the maximum over $x \in H$ we have $w(A + B) \leq \|A\| + \|B\|$. \square

Theorem 4.13. Let $B \in N_T(H)$, then $\alpha\|B\|^2 \leq w(B^2) + \frac{z^2}{2}\beta\|B\|^2$.

Proof. Without loss of generality, it is clear that an operator B is (α, β) -normal if $\alpha^2 B^* B \leq B B^* \leq \beta^2 B^* B$. Therefore, from Theorem 4.10 and use of this reverse Cauchy-Schwarz inequality $(0 \leq) \|a\| \|b\| - |\langle a, b \rangle| \leq \|a\| \|b\| - \operatorname{Re} \langle a, b \rangle \leq \frac{z^2}{2} \|a\| \|b\|$, for all $a, b \in \mathbb{C} \setminus \{0\}$ and $z > 0$. By letting $Bx = a$, $B^*x = b$ we have $\|Bx\| \|B^*x\| - |\langle Bx, B^*x \rangle| \leq \frac{z^2}{2} \|Bx\| \|B^*x\|$, and this implies that $\alpha\|Bx\|^2 \leq |\langle Bx, B^*x \rangle| + \frac{z^2}{2} \|Bx\|^2$.

Taking the maximum over $x \in H$, $\|x\| = 1$, we get $\alpha\|B\|^2 \leq w(B^2) + \frac{z^2}{2} \beta\|B\|^2$. \square

4.3 Norm inequalities

In this section we determined the norm inequalities for operators in $N_T(H)$.

Proposition 4.14. Let $A \in N_T(H)$, and $A + A^*$ be self adjoint then $\|A + A^*\| \leq \|A\| + \|A^2\|^{\frac{1}{2}} \leq 2\|A\|$.

Proof. Since $A + A^*$ is self adjoint, we have

$$\begin{aligned} \frac{1}{2}\|A + A^*\| &= \frac{1}{2}w((A + A^*) \oplus (A + A^*)) \\ &\leq w(A \oplus A^*) \\ &\leq \frac{1}{2}w(\|A \oplus A^*\| + (A \oplus (A^*)^2)^{\frac{1}{2}}) \\ &= \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \\ &\leq \|A\|. \end{aligned}$$

Hence, $\|A + A^*\| \leq \|A\| + \|A^2\|^{\frac{1}{2}} \leq 2\|A\|$. \square

Theorem 4.15. Let $A \in N_T(H)$. If A has cartesian decomposition $A = A_1 + iA_2$ such that A_1 is positive and $A_2 = A_2^*$ then $\|A^*A - AA^*\| \leq \|A_1\|^2 + \|A_2\|^2$.

Proof. By [32, Corollary 2.10] we have $\|A^*A - AA^*\| \leq \|A_1\|^2 + \|A_2\|^2$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle Ax, x \rangle|^2 &= \langle A_1x, x \rangle^2 + \langle A_2x, x \rangle^2 \\ &= \|A_1x\|^2 + \|A_2x\|^2 \\ &= \langle A_1^2x, x \rangle + \langle A_2^2x, x \rangle \\ &= \langle (A_1^2 + A_2^2)x, x \rangle. \end{aligned}$$

By analogous computation from [22, Theorem 1] we have

$$\begin{aligned} (w(A))^2 &= \sup\{|\langle Ax, x \rangle|^2 : x \in H, \|x\| = 1\} \\ &\leq \sup\{|\langle (A_1^2 + A_2^2)x, x \rangle| : x \in H, \|x\| = 1\} \\ &= \|A_1^2 + A_2^2\| \\ &= \frac{1}{2}\|A^*A + AA^*\|. \end{aligned}$$

Hence, $\|A^*A - AA^*\| \leq \|A_1\|^2 + \|A_2\|^2 = 2\|A_1A_2 - A_2A_1\| \leq \|A_1\|^2 + \|A_2\|^2$. \square

Theorem 4.16. Let $A \in N_T(H)$. Then $\alpha^r + \beta^2\|A\|^2 \leq 2\beta^r w(A^2) + r^2\beta^{2r-2}\|A - A^*\|^2$ if $r \geq 1$.

Proof. Using the Goldstein, Ryff and Clarkes [4, Theorem 2.1] inequality for vectors in inner product spaces. $\|a\|^{2r} + \|b\|^{2r} - 2\|a\|^r \|b\|^r \frac{\operatorname{Re}\langle a, b \rangle}{\|a\| \|b\|} \leq r^2 \|a\|^{2r-2} \|a - b\|^2$ if $r \geq 1$ for all $r \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \geq \|b\|$. Suppose that $r \geq 1$, let $x \in H$ with $\|x\| = 1$. Take $a = \beta Ax$, $b = A^*x$ we get $\|\beta Ax\|^{2r} + \|A^*x\|^{2r} - 2\|\beta Ax\|^{r-1} \|A^*x\|^{r-1} \operatorname{Re}\langle \beta Ax, A^*x \rangle \leq r^2 \|\beta Ax\|^{2r-2} \|\beta Ax - A^*x\|^2$ and $\alpha^{2r} + \beta^{2r} \|A\|^2 \leq 2\beta^r \|Ax\|^{r-1} \|A^2x\|^{r-1} |\langle A^2x, x \rangle| + r^2 \beta^{2r-2} \|\beta\| \|A\|^{2r-2} \|\beta Ax - A^*x\|^2$, for all $x \in H$ with $\|x\| = 1$ and $r \geq 1$. Taking the maximum over $x \in H$ with $\|x\| = 1$, we obtain $\alpha^r + \beta^2 \|A\|^2 \leq 2\beta^r w(A^2) + r^2 \beta^{2r-2} \|\beta A - A^*\|^2$, if $r \geq 1$. \square

Theorem 4.17. Let $A \in N_T(H)$. If $q \geq 2$ then

$$(1 + \alpha^q) \|A\|^q \leq \frac{1}{4} (\|A + A^*\|^q + \|A - A^*\|^q).$$

Proof. We use the following inequality obtained by Dragomir and Sandor [4, Equation 3.3] $\|c + d\|^q + \|c - d\|^q \geq 2(\|c\|^q + \|d\|^q)$ for all $c, d \in H$ and $q \geq 2$, taking $c = Ax$ and $d = A^*x$ then $\|Ax + A^*x\|^q + \|Ax - A^*x\|^q \geq 2(\|Ax\|^q + \|A^*x\|^q)$ and

$$\| \|Ax\|^q + \|A^*x\|^q + \|Ax\|^q - \|A^*x\|^q \geq 2(\|Ax\|^q + \alpha^q \|Ax\|^q)$$

for all $x \in H, \|x\| = 1$.

Therefore, taking the maximum over $x \in H, \|x\| = 1$ we obtain

$$(1 + \alpha^q) \|A\|^q \leq \frac{1}{4} (\|A + A^*\|^q + \|A - A^*\|^q). \quad \square$$

Corollary 4.18. Let $A \in N_T(H)$. If $q \geq 2$ and $\alpha = 0$, then $\|A\|^q \leq \frac{1}{4} (\|A + A^*\|^q + \|A - A^*\|^q)$.

Proof. We give analogous prove of Theorem 4.17, using the following in-

equality in [10, Equation 3.15] i.e $\|a + b\|^q + \|a - b\|^q \geq 2(\|a\|^q + \|b\|^q)$.

For all $a, b \in H$ and $q \geq 2$, taking $a = Ax$ and $b = A^*x$ then

$\|Ax + A^*x\|^q + \|Ax - A^*x\|^q \geq 2(\|Ax\|^q + \|A^*x\|^q)$ and

$$\| \|Ax\|^q + \|A^*x\|^q \| + \|Ax\|^q - \|A^*x\|^q \geq 2(\|Ax\|^q + \|A^*x\|^q)$$

for all $x \in H, \|x\| = 1$.

Therefore, taking the maximum over $x \in H, \|x\| = 1$ we obtain

$$\|A\|^q \leq \frac{1}{4}(\|A + A^*\|^q + \|A - A^*\|^q). \quad \square$$

Theorem 4.19. Let $A \in N_T(H)$ and $\alpha, \beta \in \mathbb{C}$. Then if $q \geq 2$ we have

$$\|\alpha A + \beta A^*\|^q + \|\alpha A - \beta A^*\|^q \geq 2(|\alpha|^q + |\beta|^q)\|A\|^q.$$

Proof. By use of parallelogram identity and the known [4, Equation 3.3] inequality, $\|a + b\|^q + \|a - b\|^q \geq 2(\|a\|^q + \|b\|^q)$. Let a, b be arbitrary vectors in the vector space, such that $a = \alpha Ax, b = \beta A^*x$ we obtain $\|(\alpha A + \beta A^*)(x)\|^q + \|(\alpha A - \beta A^*)(x)\|^q \geq 2(|\alpha|^q + |\beta|^q)\|Ax\|^q$.

Taking the maximum over $x \in H, \|x\| = 1$ we obtain $\|\alpha A + \beta A^*\|^q + \|\alpha A - \beta A^*\|^q \geq 2(|\alpha|^q + |\beta|^q)\|A\|^q. \quad \square$

Remark 4.20. For the parallelogram identity, if $q = 2$ we obtain, $\|\alpha A + \beta A^*\|^2 + \|\alpha A - \beta A^*\|^2 \geq 2(|\alpha|^2 + |\beta|^2)\|A\|^2$.

Theorem 4.21. Let $A \in N_T(H)$. If $q \geq 1$ and $|\alpha| \geq 1$ then

$$(1 - |\alpha|^q)^2\|A\|^2 \leq q^2\|A - \alpha A^*\|^2.$$

Proof. Consider [10, Equation 3.8] and $\|a\|^{2q} + \|b\|^{2q} - 2\|a\|^{q-1}\|b\|^{q-1}|\langle a, b \rangle| \leq q^2\|a\|^{2q-2}\|a - b\|^2$ if $q \geq 1$ for all $q \in \mathbb{R}, a, b \in H$ with $\|a\| \geq \|b\|$. Choose $a = Ax, b = \alpha A^*x$ and since $\|Ax\| = \|A^*x\|$ we have

$$\|Ax\|^{2q} + |\alpha|^{2q} \|Ax\|^{2q} \leq 2\|Ax\|^{2q-2} |\alpha|^q |\langle A^2x, x \rangle| + q^2 \|Ax\|^{2q-2} \|Ax - \alpha A^*x\|^2$$

and this implies that $(1 + |\alpha|^{2q}) \|Ax\|^{2q} \leq 2|\alpha|^q |\langle A^2x, x \rangle| + q^2 \|Ax - \alpha A^*x\|^2$.

Taking the maximum over $x \in H$, $\|x\| = 1$, we obtain $(1 - |\alpha|^q)^2 \|A\|^2 \leq q^2 \|A - \alpha A^*\|^2$. □

CONCL

RECOM

5.1. Introduction

In this chapter, we draw
 certain objectives of the

5.3. Conclusion

We summarize the work
 in this study in respect to
 the various theorems
 of the study has been de
 scribed with preliminary
 work in the field of the
 study of the study

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

In this chapter, we draw conclusions and make recommendations based on our objectives of the study and the results obtained.

5.2 Conclusions

We summarize our work by highlighting the results we have obtained in our study in respect to the problem stated in Section 1.3. Chapter one contains the introduction of the study in which good background of the study has been discussed in details and also contains basic concepts with preliminary results which form a prerequisite to the main work in this thesis. In the first objective we have investigated the properties of (α, β) - normal transaloid operators and established the follow-

ing properties for the class of $N_T(H)$: Linearity, Continuity, Positivity, boundedness, Reflexivity and Self-adjointness. These properties have been discussed in Chapter Four where results on (α, β) - normal transaloid operators are given in details. Also, in Chapter Four we have obtained the results on numerical radii inequalities and norm inequalities for (α, β) - normal transaloid operators. On numerical radii inequalities, the following results have been obtained. Let $A \in N_T(H)$. If A is self-adjoint then $w(A^*) = \frac{\|A^*\| + \|(A^*)^2\|^{\frac{1}{2}}}{2} = \|A\|$, also if $A \in N_T(H)$ then $w(A) = \frac{1}{2}(\|A\| + \|(A^2)\|^{\frac{1}{2}})$. Let $A, B \in N_T(H)$ then $w(AB) \leq 2w(A)(B)$ if $AB = BA$, Let $A_1, A_2, A_3, A_4 \in N_T(H)$ and $B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ then

$$\max(w(A_1), w(A_4)) \leq \frac{1}{2}(\|B\| + \|B^2\|^{\frac{1}{2}})$$

also

$$\max(w(A_2A_3)^{\frac{1}{2}}, w(A_3A_4)^{\frac{1}{2}}) \leq \frac{1}{2}(\|B\| + \|B^2\|^{\frac{1}{2}}).$$

If $B \in N_T(H)$, then $\alpha\|B\|^2 \leq w(B^2) + \frac{\alpha^2}{2}\beta\|B\|^2$. Let A and B be self-adjoint operators in $N_T(H)$, and $q \geq 2$. Then $w^q(A+B) \leq 2^{q-1}(\|A\|^q + \|B\|^q)$. If $A \in N_T(H)$, then $w(A)^2 \leq \frac{1}{2}[\alpha\|A\|^2 + w(A^2)]$. Similarly, we have determined the results for the norm inequalities for this class of $N_T(H)$. If $A \in N_T(H)$, then $\|A + A^*\| \leq \|A\| + \|A^2\|^{\frac{1}{2}} \leq 2\|A\|$. Also if $A \in N_T(H)$ and A has cartesian decomposition $A = A_1 + iA_2$ such that A_1 is positive and $A_2 = A_2^*$ then $\|A^*A - AA^*\| \leq \|A_1\|^2 + \|A_2\|^2$. If $A \in N_T(H)$, then $\alpha^r + \beta^2\|A\|^2 \leq 2\beta^r w(A^2) + r^2\beta^{2r-2}\|\beta A - A^*\|^2$ if $r \geq 1$. Let $A \in N_T(H)$ and $\alpha, \beta \in \mathbb{C}$ then if $q \geq 2$ we have $\|\alpha A + \beta A^*\|^q + \|\alpha A - \beta A^*\|^q \geq 2(|\alpha|^q + |\beta|^q)\|A\|^q$. Let $A \in$

$N_T(H)$, if $q \geq 1$ and $|\alpha| \geq 1$ then $(1 - |\alpha|^q)^2 \|A\|^2 \leq q^2 \|A - \alpha A^*\|^2$. Lastly, if $A \in N_T(H)$, if $q \geq 2$ then $(1 + \alpha^q) \|A\|^q \leq \frac{1}{4} (\|A + A^*\|^q + \|A - A^*\|^q)$. Finally, the results we have obtained may be applicable in quantum theory, particularly in commutator approximation and estimation of ground state energies.

5.3 Recommendations

Norms and numerical radii inequalities for (α, β) -normal transaloid operators is an area with interesting results. Absolutely, the properties and the inequalities we have established in Chapter Four provides a platform and an avenue for all the researchers to further their research on other classes of operator theory. We recommend that further investigation should be done to establish whether the results so far obtained in this study for (α, β) -normal transaloid operators can be applied to generalized self-adjoint property for (α, β) -normal transaloid operators.

Moreover, further research can be directed on investigating the structural properties of the algebra of all (α, β) -normal transaloid operators as this will be useful in determining norms of these operators.

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